2 Trigonometry

BUILDING ON

- applying the Pythagorean Theorem
- solving problems using properties of similar polygons
- solving problems involving ratios

BIG IDEAS

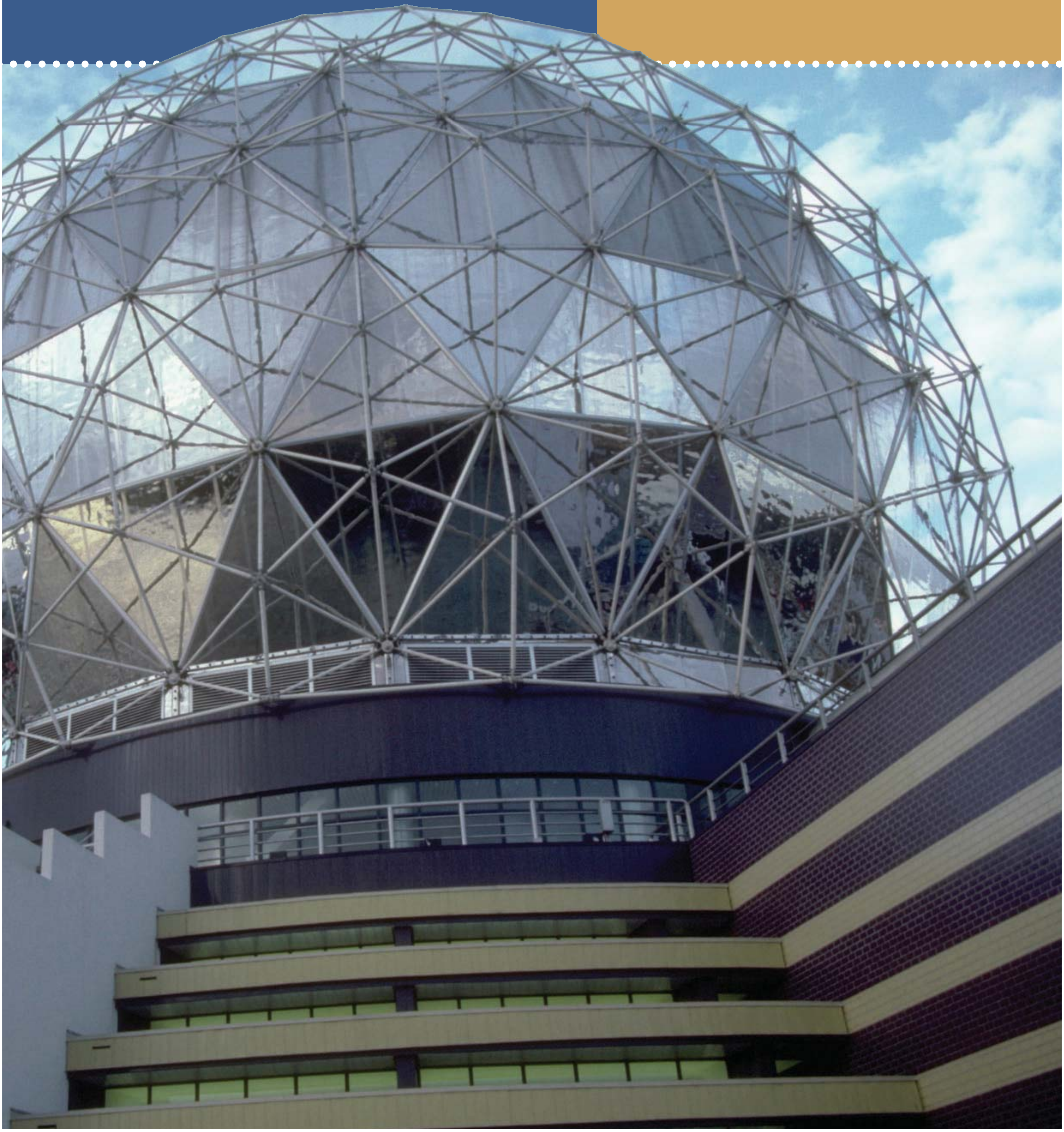
In a right triangle,

- The ratio of any two sides remains constant even if the triangle is enlarged or reduced.
- You can use the ratio of the lengths of two sides to determine the measure of one of the acute angles.
- You can use the length of one side and the measure of an acute angle to determine the length of another side of the triangle.

NEW VOCABULARY

angle of inclination
tangent ratio
indirect measurement
sine ratio
cosine ratio
angle of elevation
angle of depression

SCIENCE WORLD *This building was constructed for the Expo '86 World Fair held in Vancouver, British Columbia. The structure is a geodesic dome containing 766 triangles.*



2.1 The Tangent Ratio

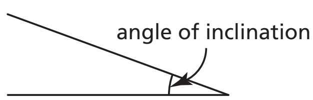
LESSON FOCUS

Develop the tangent ratio and relate it to the angle of inclination of a line segment.

This ranger's cabin on Herschel Island, Yukon, has solar panels on its roof.

Make Connections

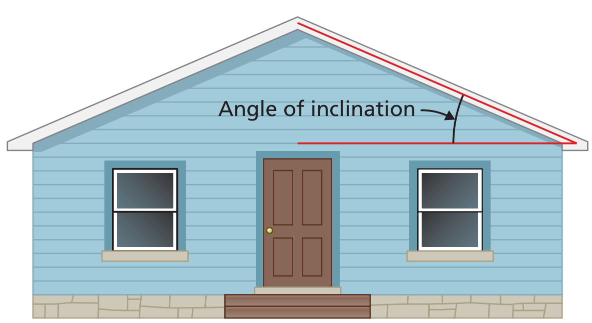
The **angle of inclination** of a line or line segment is the acute angle it makes with the horizontal.



South-facing solar panels on a roof work best when the **angle of inclination** of the roof, that is, the angle between the roof and the horizontal, is approximately equal to the latitude of the house.

When an architect designs a house that will have solar panels on its roof, she has to determine the width and height of the roof so that the panels work efficiently.

What happens to the angle of inclination if the diagram of the house is drawn using a different scale?



You will investigate the relationship between one acute angle in a right triangle and two sides of that triangle.

Construct Understanding

Recall that two triangles are similar if one triangle is an enlargement or a reduction of the other.

TRY THIS

Work with a partner.

You will need grid paper, a ruler, and a protractor.

A. On grid paper, draw a right $\triangle ABC$ with $\angle B = 90°$.

B. Each of you draws a different right triangle that is similar to $\triangle ABC$.

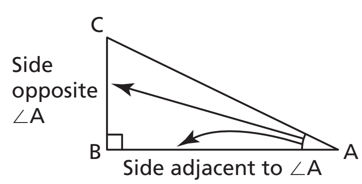
C. Measure the sides and angles of each triangle. Label your diagrams with the measures.

D. The two shorter sides of a right triangle are its legs. Calculate the ratio of the legs $\frac{CB}{BA}$ as a decimal, then the corresponding ratio for each of the similar triangles.

E. How do the ratios compare?

F. What do you think the value of each ratio depends on?

We name the sides of a right triangle in relation to one of its acute angles.



The ratio

Length of side opposite $\angle A$: Length of side adjacent to $\angle A$

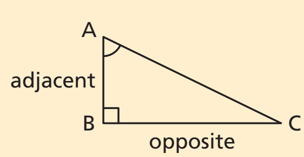
depends only on the measure of the angle, not on how large or small the triangle is.

This ratio is called the **tangent ratio** of $\angle A$.
The tangent ratio for $\angle A$ is written as tan A.
We usually write the tangent ratio as a fraction.

The Tangent Ratio

If $\angle A$ is an acute angle in a right triangle, then

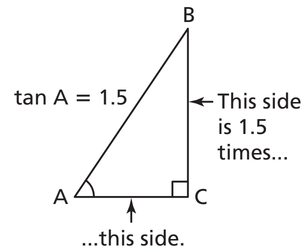
$$\tan A = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A}$$



As the size of ∠A increases, what happens to tan A?

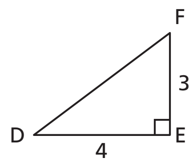
The value of the tangent ratio is usually expressed as a decimal that compares the lengths of the sides.

For example, if tan A = 1.5; then, in any similar right triangle with ∠A, the length of the side opposite ∠A is 1.5 times the length of the side adjacent to ∠A.



Example 1 Determining the Tangent Ratios for Angles

Determine tan D and tan F.

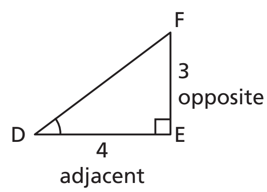


SOLUTION

$$\tan D = \frac{\text{length of side opposite } \angle D}{\text{length of side adjacent to } \angle D}$$

$$\tan D = \frac{EF}{DE}$$

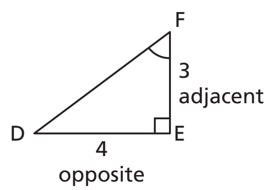
EF is opposite ∠D, DE is adjacent to ∠D.



$$\tan D = \frac{3}{4}$$

$$\tan D = 0.75$$

$$\tan F = \frac{\text{length of side opposite } \angle F}{\text{length of side adjacent to } \angle F}$$



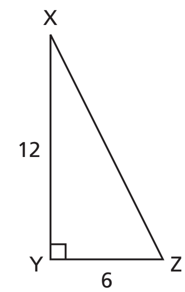
$$\tan F = \frac{DE}{EF}$$

$$\tan F = \frac{4}{3}$$

$$\tan F = 1.\bar{3}$$

CHECK YOUR UNDERSTANDING

1. Determine tan X and tan Z.



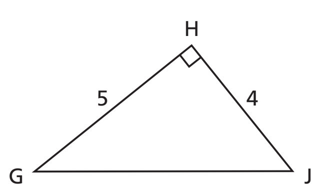
[Answer: tan X = 0.5; tan Z = 2]

How are the values of tan D and tan F related? Explain why this relation will always be true for the acute angles in a right triangle.

You can use a scientific calculator to determine the measure of an acute angle when you know the value of its tangent. The $\tan^{-1}$ or InvTan calculator operation does this.

Example 2 Using the Tangent Ratio to Determine the Measure of an Angle

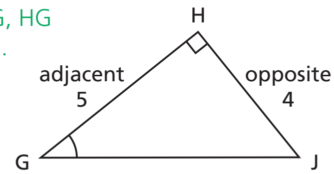
Determine the measures of ∠G and ∠J to the nearest tenth of a degree.



SOLUTION

In right △GHJ:

HJ is opposite ∠G, HG is adjacent to ∠G.



$$\tan G = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan G = \frac{HJ}{GH}$$

$$\tan G = \frac{4}{5}$$

$$\angle G \doteq 38.7°$$

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tan⁻¹(0.8)
38.65980825
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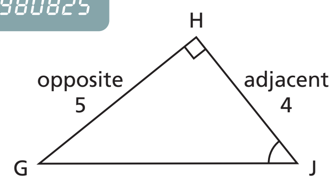

$$\tan J = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan J = \frac{GH}{HJ}$$

$$\tan J = \frac{5}{4}$$

$$\angle J \doteq 51.3°$$

```
tan⁻¹(1.25)
51.34019175
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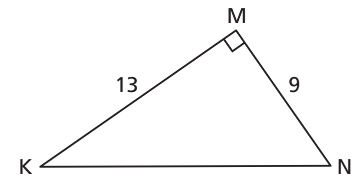
CHECK YOUR UNDERSTANDING

2. Determine the measures of ∠K and ∠N to the nearest tenth of a degree.

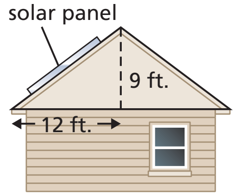


[Answer: ∠K ≐ 34.7°; ∠N ≐ 55.3°]

What other strategy could you use to determine ∠J?

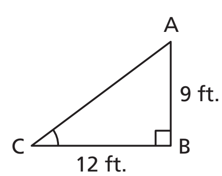
Example 3 Using the Tangent Ratio to Determine an Angle of Inclination

The latitude of Fort Smith, NWT, is approximately 60°. Determine whether this design for a solar panel is the best for Fort Smith. Justify your answer.



SOLUTION

The best angle of inclination for the solar panel is the same as the latitude, 60°. Draw a right triangle to represent the cross-section of the roof and solar panel. ∠C is the angle of inclination. In △ABC:



$$\tan C = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan C = \frac{AB}{BC}$$

AB is opposite ∠C, BC is adjacent to ∠C.

$$\tan C = \frac{9}{12}$$

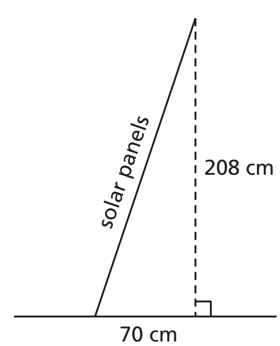
$$\angle C \doteq 37°$$

```
tan⁻¹(0.75)
36.86989765
```

The angle of inclination of the solar panel is about 37°, which is not equal to the latitude of Fort Smith. So, this is not the best design.

CHECK YOUR UNDERSTANDING

3. Clyde River on Baffin Island, Nunavut, has a latitude of approximately 70°. The diagram shows the side view of some solar panels. Determine whether this design for solar panels is the best for Clyde River. Justify your answer.



[Answer: The angle of inclination is approximately 71°. So, the design is the best.]

Example 4 Using the Tangent Ratio to Solve a Problem

A 10-ft. ladder leans against the side of a building with its base 4 ft. from the wall.

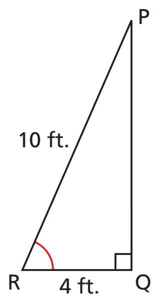
What angle, to the nearest degree, does the ladder make with the ground?

SOLUTION

Draw a diagram.

Label the vertices of the triangle PQR.

Assume the ground is horizontal and the building vertical.



To use the tangent ratio to determine $\angle R$, we first need to know the length of PQ.

Use the Pythagorean Theorem in right $\triangle PQR$.

$$PR^2 = PQ^2 + QR^2$$

Isolate the unknown.

$$PQ^2 = PR^2 - QR^2$$

$$PQ^2 = 10^2 - 4^2$$

$$= 84$$

$$PQ = \sqrt{84}$$

Use the tangent ratio in right $\triangle PQR$.

$$\tan R = \frac{PQ}{QR}$$

PQ is opposite $\angle R$, QR is adjacent to $\angle R$.

$$\tan R = \frac{\sqrt{84}}{4}$$

$$\tan R = 2.2913\ldots$$

$$\angle R \doteq 66°$$

The angle between the ladder and the ground is approximately 66°.

CHECK YOUR UNDERSTANDING

4. A support cable is anchored to the ground 5 m from the base of a telephone pole. The cable is 19 m long. It is attached near the top of the pole. What angle, to the nearest degree, does the cable make with the ground?

[Answer: The angle is approximately 75°.]

Suppose you used $PQ \doteq 9.2$, instead of $PQ = \sqrt{84}$. How could this affect the calculated measure of $\angle R$?

Discuss the Ideas

1. Why does the value of the tangent ratio of a given angle not depend on the right triangle you use to calculate it?

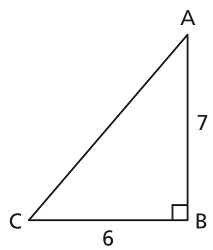
2. How can you use the tangent ratio to determine the measures of the acute angles of a right triangle when you know the lengths of its legs?

Exercises

A

3. In each triangle, write the tangent ratio for each acute angle.

a)



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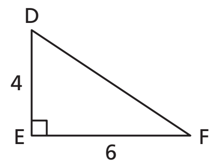
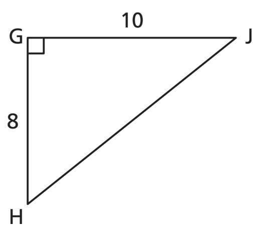
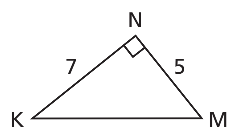


c)



d)



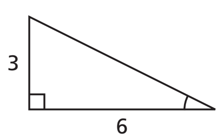
4. To the nearest degree, determine the measure of ∠X for each value of tan X.

a) $\tan X = 0.25$ **b)** $\tan X = 1.25$

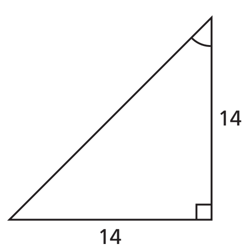
c) $\tan X = 2.50$ **d)** $\tan X = 20$

5. Determine the measure of each indicated angle to the nearest degree.

a)



b)



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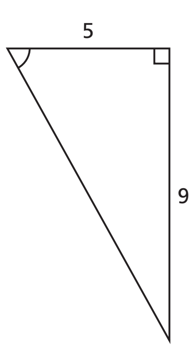

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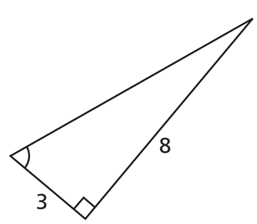


B

6. Use grid paper. Illustrate each tangent ratio by sketching a right triangle, then labelling the measures of its legs.

a) $\tan B = \frac{3}{5}$ **b)** $\tan E = \frac{5}{3}$ **c)** $\tan F = \frac{1}{4}$

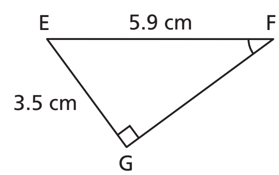
d) $\tan G = 4$ **e)** $\tan H = 1$ **f)** $\tan J = 25$

7. a) Is tan 60° greater than or less than 1? How do you know without using a calculator?

b) Is tan 30° greater than or less than 1? How do you know without using a calculator?

8. Determine the measure of each indicated angle to the nearest tenth of a degree. Describe your solution method.

a)



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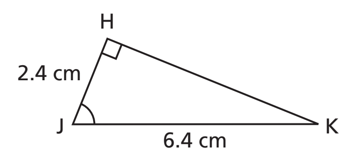
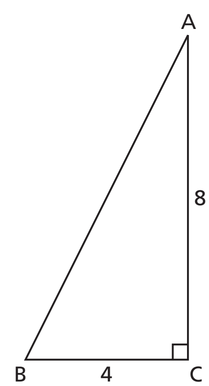
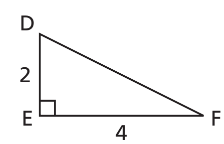


9. a) Why are these triangles similar?

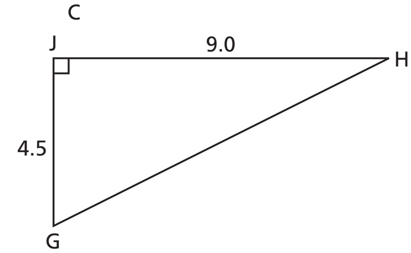
i)



ii)



iii)

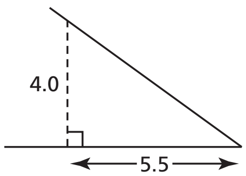


b) For each triangle in part a, determine the measures of the acute angles to the nearest tenth of a degree.

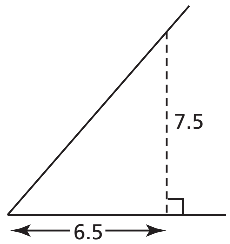
c) To complete part b, did you have to calculate the measures of all 6 acute angles? Explain.

10. Determine the angle of inclination of each line to the nearest tenth of a degree.

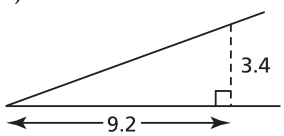
a)



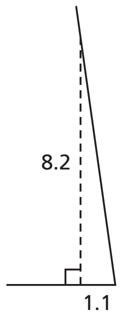
b)



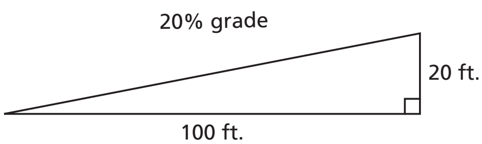
c)



d)



11. The grade or inclination of a road is often expressed as a percent. When a road has a grade of 20%, it increases 20 ft. in altitude for every 100 ft. of horizontal distance.



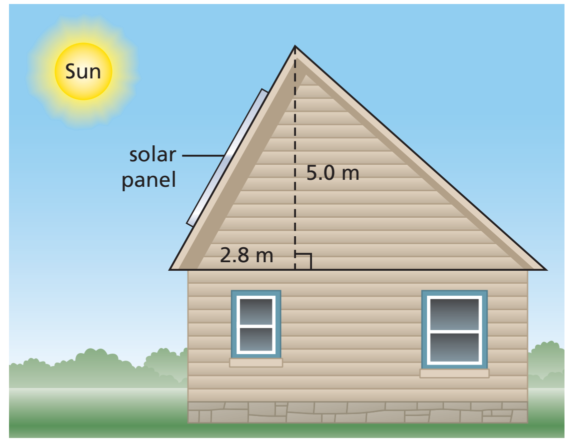
Calculate the angle of inclination, to the nearest degree, of a road with each grade.

a) 20% **b)** 25% **c)** 10% **d)** 15%

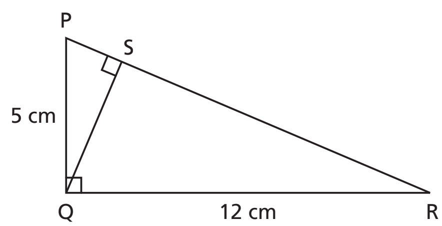
12. The approximate latitudes for several cities in western and northern Canada are shown.

Calgary	51.1°	Edmonton	53.5°
Fort McMurray	56.5°	Inuvik	68.4°
Kamloops	51.2°	Saskatoon	52.2°
Victoria	48.4°	Whitehorse	60.7°

For which locations might the following roof design be within 1° of the recommended angle for solar panels? Justify your answer.



13. Determine the measures of all the acute angles in this diagram, to the nearest tenth of a degree.



14. A birdwatcher sights an eagle at the top of a 20-m tree. The birdwatcher is lying on the ground 50 m from the tree. At what angle must he incline his camera to take a photograph of the eagle? Give the answer to the nearest degree.

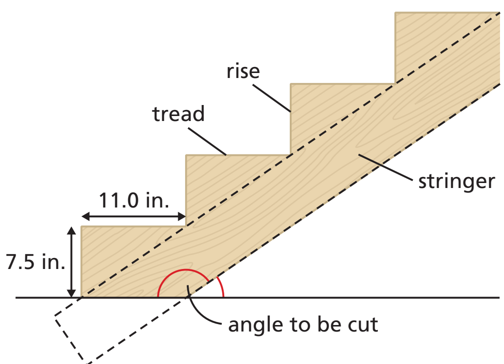
15. A rectangle has dimensions 3 cm by 8 cm. What angles does a diagonal of the rectangle make with the sides of the rectangle? Give the measures to the nearest tenth of a degree.

16. In a right isosceles triangle, why is the tangent of an acute angle equal to 1?

17. A playground slide starts 107 cm above the ground and is 250 cm long. What angle does the slide make with the ground? Give the answer to the nearest degree.

18. The Pioneer ski lift at Golden, B.C., is 1366 m long. It rises 522 m vertically. What is the angle of inclination of the ski lift? Give the answer to the nearest degree.

19. From a rectangular board, a carpenter cuts a stringer to support some stairs. Each stair rises 7.5 in. and has a tread of 11.0 in. To the nearest degree, at what angle should the carpenter cut the board?



20. For safety reasons, a ladder is positioned so that the distance between its base and the wall is no greater than $\frac{1}{4}$ the length of the ladder. To the nearest degree, what is the greatest angle of inclination allowed for a ladder?

C

21. In isosceles △XYZ, XY = XZ = 5.9 cm and YZ = 5.0 cm. Determine the measures of the angles of the triangle to the nearest tenth of a degree.

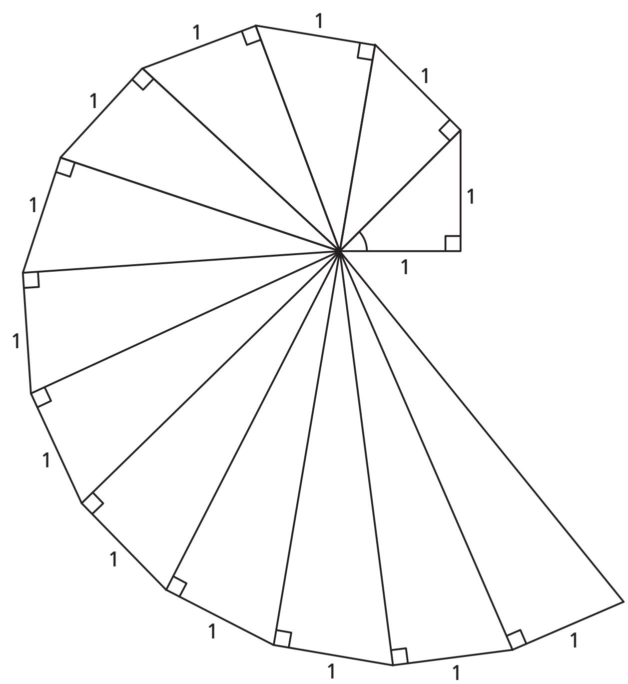
22. For the tangent of an acute angle in a right triangle:

a) What is the least possible value?

b) What is the greatest possible value?

Justify your answers.

23. A Pythagorean spiral is constructed by drawing right triangles on the hypotenuse of other right triangles. Start with a right triangle in which each leg is 1 unit long. Use the hypotenuse of that triangle as one leg of a new triangle and draw the other leg 1 unit long. Continue the process. A spiral is formed.



a) Determine the tangent of the angle at the centre of the spiral in each of the first 5 triangles.

b) Use the pattern in part a to predict the tangent of the angle at the centre of the spiral for the 100th triangle. Justify your answer.

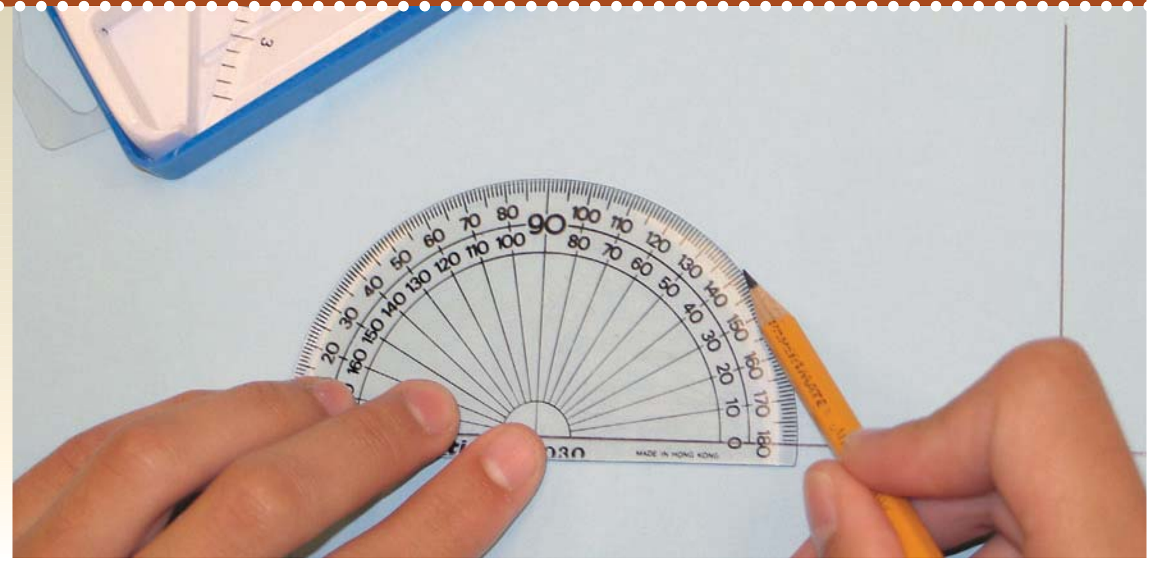
Reflect

Summarize what you have learned about the tangent ratio and its relationship to the sides and angles of a right triangle.

2.2 Using the Tangent Ratio to Calculate Lengths

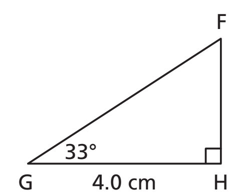
LESSON FOCUS

Apply the tangent ratio to calculate lengths.



Make Connections

In Lesson 2.1, you used the measures of two legs of a right triangle to calculate the measures of the acute angles of the triangle. When you know the length of one leg of a right triangle and the measure of one acute angle, you can draw the triangle.



What other measures in the triangle can you calculate?

Construct Understanding

THINK ABOUT IT

Work with a partner.

In right △PQR, ∠Q = 90°, ∠P = 34.5°, and PQ = 46.1 cm.
Determine the length of RQ to the nearest tenth of a centimetre.

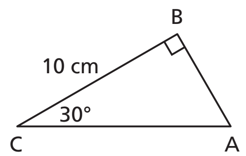
We use **direct measurement** when we use a measuring instrument to determine a length or an angle in a polygon. We use **indirect measurement** when we use mathematical reasoning to calculate a length or an angle.

The tangent ratio is a powerful tool we can use to calculate the length of a leg of a right triangle. We are then measuring the length of a side of a triangle **indirectly**.

In a right triangle, we can use the tangent ratio, $\frac{\text{opposite}}{\text{adjacent}}$, to write an equation. When we know the measure of an acute angle and the length of a leg, we solve the equation to determine the length of the other leg.

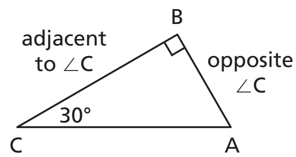
Example 1 Determining the Length of a Side Opposite a Given Angle

Determine the length of AB to the nearest tenth of a centimetre.



SOLUTION

In right △ABC, AB is the side opposite ∠C and BC is the side adjacent to ∠C.



Use the tangent ratio to write an equation.

$$\tan C = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan C = \frac{AB}{BC}$$

$$\tan 30° = \frac{AB}{10}$$

Solve this equation for AB.

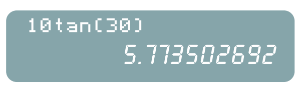
Multiply both sides by 10.

$$10 \times \tan 30° = \frac{AB}{10} \times 10$$

$$10 \tan 30° = AB$$

We write: 10 × tan 30° as 10 tan 30°
When an operation sign is omitted, it is understood to be multiplication.

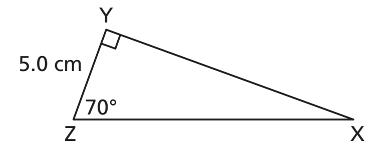
$$AB = 5.7735\ldots$$



AB is approximately 5.8 cm long.

CHECK YOUR UNDERSTANDING

1. Determine the length of XY to the nearest tenth of a centimetre.

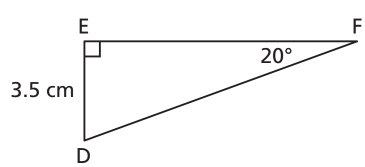


[Answer: XY ≐ 13.7 cm]

How can you determine the length of the hypotenuse in △ABC?

Example 2 Determining the Length of a Side Adjacent to a Given Angle

Determine the length of EF to the nearest tenth of a centimetre.



SOLUTIONS

Method 1

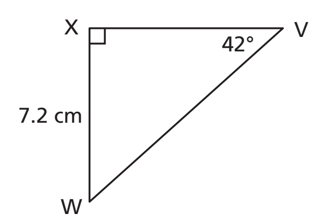
In right △DEF, DE is opposite ∠F and EF is adjacent to ∠F.

$$\tan F = \frac{\text{opposite}}{\text{adjacent}}$$

(Solution continues.)

CHECK YOUR UNDERSTANDING

2. Determine the length of VX to the nearest tenth of a centimetre.



[Answer: VX ≐ 8.0 cm]

$$\tan F = \frac{DE}{EF}$$

$$\tan 20° = \frac{3.5}{EF}$$

Solve the equation for EF. Multiply both sides by EF.

$$EF \tan 20° = EF\left(\frac{3.5}{EF}\right)$$

$$EF \tan 20° = 3.5$$ Divide both sides by tan 20°.

$$\frac{EF \tan 20°}{\tan 20°} = \frac{3.5}{\tan 20°}$$

$$EF = \frac{3.5}{\tan 20°}$$

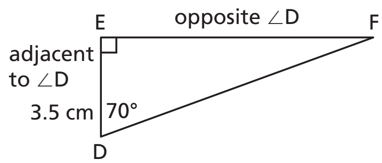
$$EF = 9.6161\ldots$$

EF is approximately 9.6 cm long.

What is the advantage of solving the equation for EF before calculating tan 20°?

Method 2

In right △DEF:



$$\angle D + \angle F = 90°$$

$$\angle D + 20° = 90°$$

$$\angle D = 90° - 20°$$

$$\angle D = 70°$$

EF is opposite ∠D and DE is adjacent to ∠D.

$$\tan D = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan D = \frac{EF}{DE}$$

$$\tan 70° = \frac{EF}{3.5}$$

Solve the equation for EF. Multiply both sides by 3.5.

$$3.5 \tan 70° = \frac{(EF)(3.5)}{3.5}$$

$$3.5 \tan 70° = EF$$

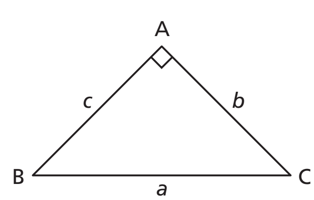
$$EF = 9.6161\ldots$$

EF is approximately 9.6 cm long.

Which method to determine EF do you think is easier? Why?

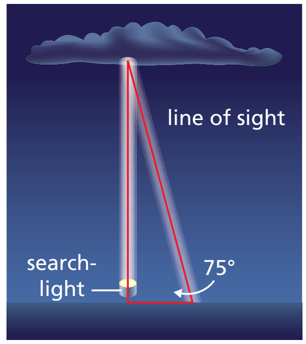
How could you determine the length of DF?

It is often convenient to use the lower case letter to name the side opposite a vertex of a triangle.



Example 3 Using Tangent to Solve an Indirect Measurement Problem

A searchlight beam shines vertically on a cloud. At a horizontal distance of 250 m from the searchlight, the angle between the ground and the line of sight to the cloud is 75°. Determine the height of the cloud to the nearest metre.

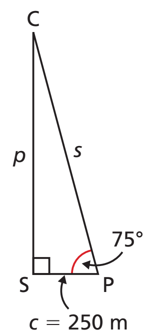


SOLUTION

Sketch and label a diagram to represent the information in the problem.

Assume the ground is horizontal.

In right △CSP, side CS is opposite ∠P and SP is adjacent to ∠P.



$$\tan P = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan P = \frac{p}{c}$$

$$\tan 75° = \frac{p}{250}$$

Solve the equation for p. Multiply both sides by 250.

$$250 \tan 75° = \left(\frac{p}{250}\right)250$$

$$250 \tan 75° = p$$

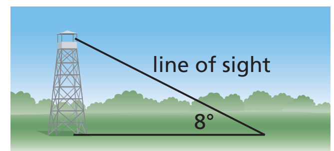
$$p = 933.0127\ldots$$

The cloud is approximately 933 m high.

Why can we draw a right triangle to represent the problem?

CHECK YOUR UNDERSTANDING

3. At a horizontal distance of 200 m from the base of an observation tower, the angle between the ground and the line of sight to the top of the tower is 8°. How high is the tower to the nearest metre? The diagram is *not* drawn to scale.



[Answer: 28 m]

Discuss the Ideas

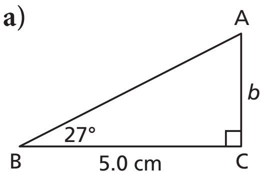
1. How can you use the tangent ratio to determine the length of a leg in a right triangle?

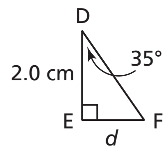
2. Suppose you know or can calculate the lengths of the legs in a right triangle. Why can you always calculate its hypotenuse?

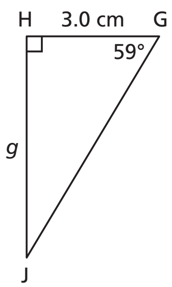
Exercises

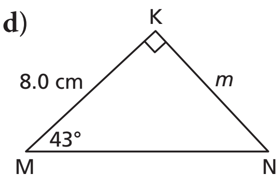
A

3. Determine the length of each indicated side to the nearest tenth of a centimetre.

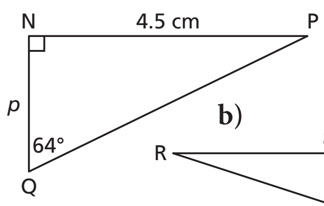
a)


b)


c)


d)


4. Determine the length of each indicated side to the nearest tenth of a centimetre.

a)


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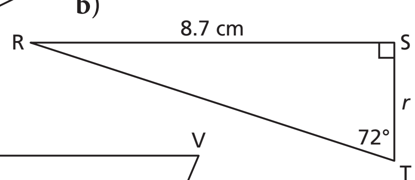

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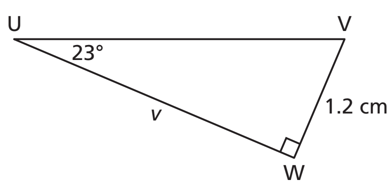
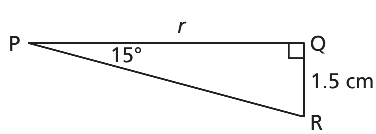


5. Determine the length of each indicated side to the nearest tenth of a centimetre.

a)


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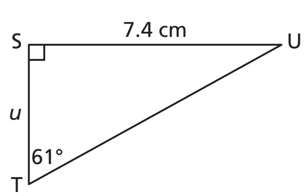
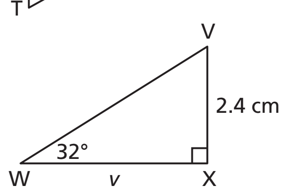
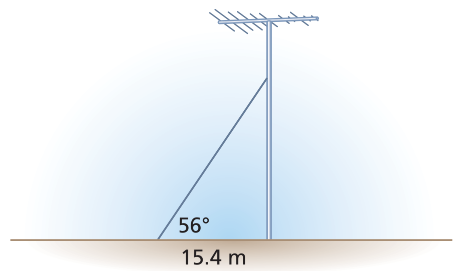


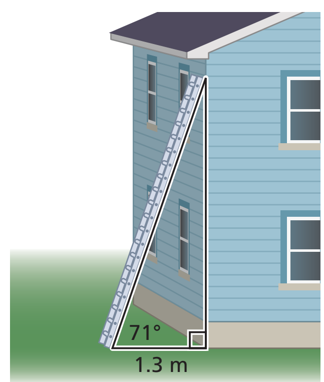
c)


B

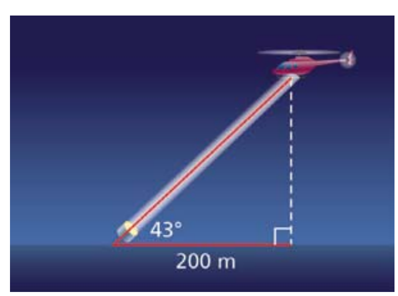
6. A guy wire helps to support a tower. The angle between the wire and the level ground is 56°. One end of the wire is 15.4 m from the base of the tower. How high up the tower does the wire reach to the nearest tenth of a metre?



7. The base of a ladder is on level ground 1.3 m from a wall. The ladder leans against the wall. The angle between the ladder and the ground is 71°. How far up the wall does the ladder reach to the nearest tenth of a metre?

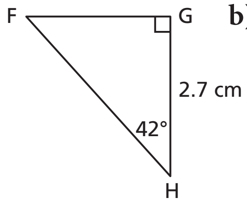


8. A helicopter is descending vertically. On the ground, a searchlight is 200 m from the point where the helicopter will land. It shines on the helicopter and the angle the beam makes with the ground is 43°. How high is the helicopter at this point to the nearest metre?

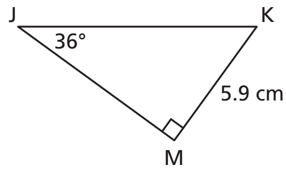


9. Determine the length of the hypotenuse of each right triangle to the nearest tenth of a centimetre. Describe your strategy.

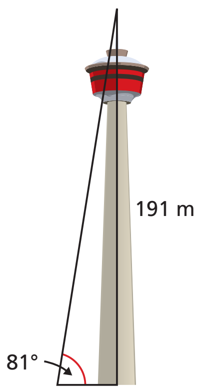
a)



b)



10. Claire knows that the Calgary Tower is 191 m high. At a certain point, the angle between the ground and Claire's line of sight to the top of the tower was 81°. To the nearest metre, about how far was Claire from the tower? Why is this distance approximate?



11. The angle between one longer side of a rectangle and a diagonal is 34°. One shorter side of the rectangle is 2.3 cm.

a) Sketch and label the rectangle.

b) What is the length of the rectangle to the nearest tenth of a centimetre?

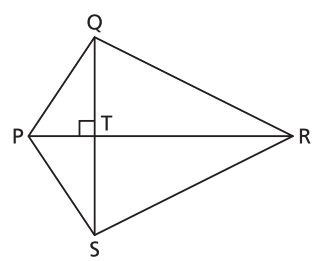
12. In $\triangle PQR$, $\angle R = 90°$, $\angle P = 58°$, and PR = 7.1 cm. Determine the area of $\triangle PQR$ to the nearest tenth of a square centimetre. Describe your strategy.

13. The height of the Manitoba Legislature Building, from the ground to the top of the Golden Boy statue, is about 77 m. Liam is lying on the ground near the building. The angle between the ground and his line of sight to the top of the building is 52°. About how far is Liam from a point on the ground vertically below the statue? How do you know?

14. Janelle sees a large helium-filled balloon anchored to the roof of a store. When she is 100 m from the store, the angle between the ground and her line of sight to the balloon is 30°. About how high is the balloon? What assumptions are you making?

C

15. In kite PQRS, the shorter diagonal, QS, is 6.0 cm long, $\angle QRT$ is 26.5°, and $\angle QPT$ is 56.3°. Determine the measures of all the angles and the lengths of the sides of the kite to the nearest tenth.



16. On a coordinate grid:

a) Draw a line through the points A(4, 5) and B(−4, −5). Determine the measure of the acute angle between AB and the y-axis.

b) Draw a line through the points C(1, 4) and D(4, −2). Determine the measure of the acute angle between CD and the x-axis.

Reflect

Summarize what you have learned about using the tangent ratio to determine the length of a side of a right triangle.

MATH LAB

2.3 Measuring an Inaccessible Height

LESSON FOCUS

Determine a height that cannot be measured directly.

Make Connections

Tree farmers use a *clinometer* to measure the angle between a horizontal line and the line of sight to the top of a tree. They measure the distance to the base of the tree. How can they then use the tangent ratio to calculate the height of the tree?

Construct Understanding

TRY THIS

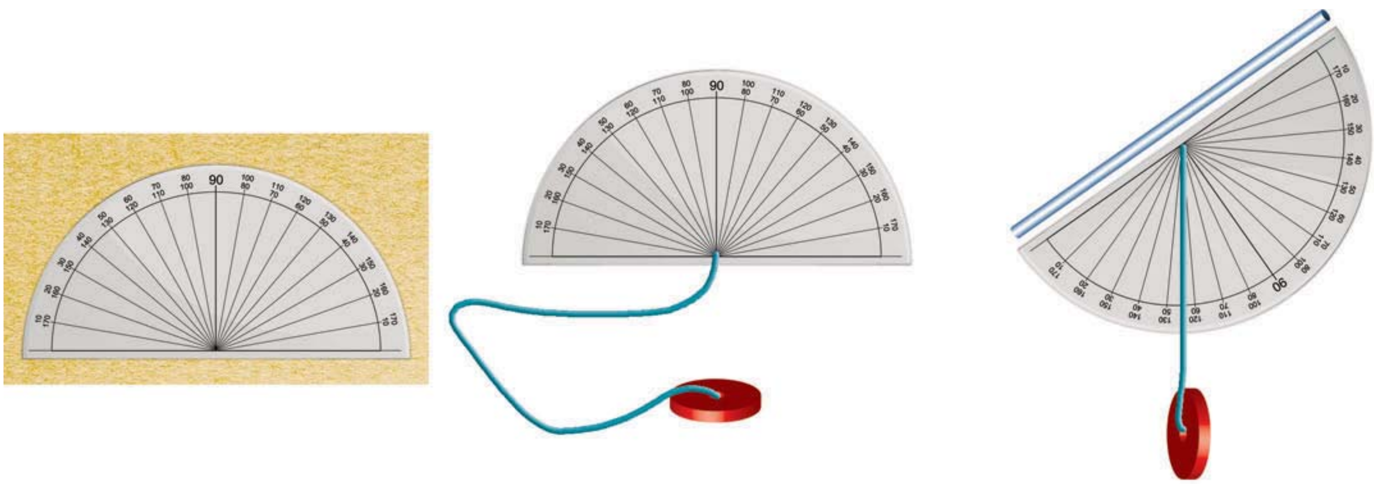
Work with a partner.

You will need:

- an enlarged copy of a 180° protractor
- scissors
- a measuring tape or 2 metre sticks
- a piece of heavy cardboard big enough for you to attach the paper protractor
- a drinking straw
- glue
- adhesive tape
- a needle and thread
- a small metal washer or weight
- grid paper

A. Make a drinking straw clinometer:

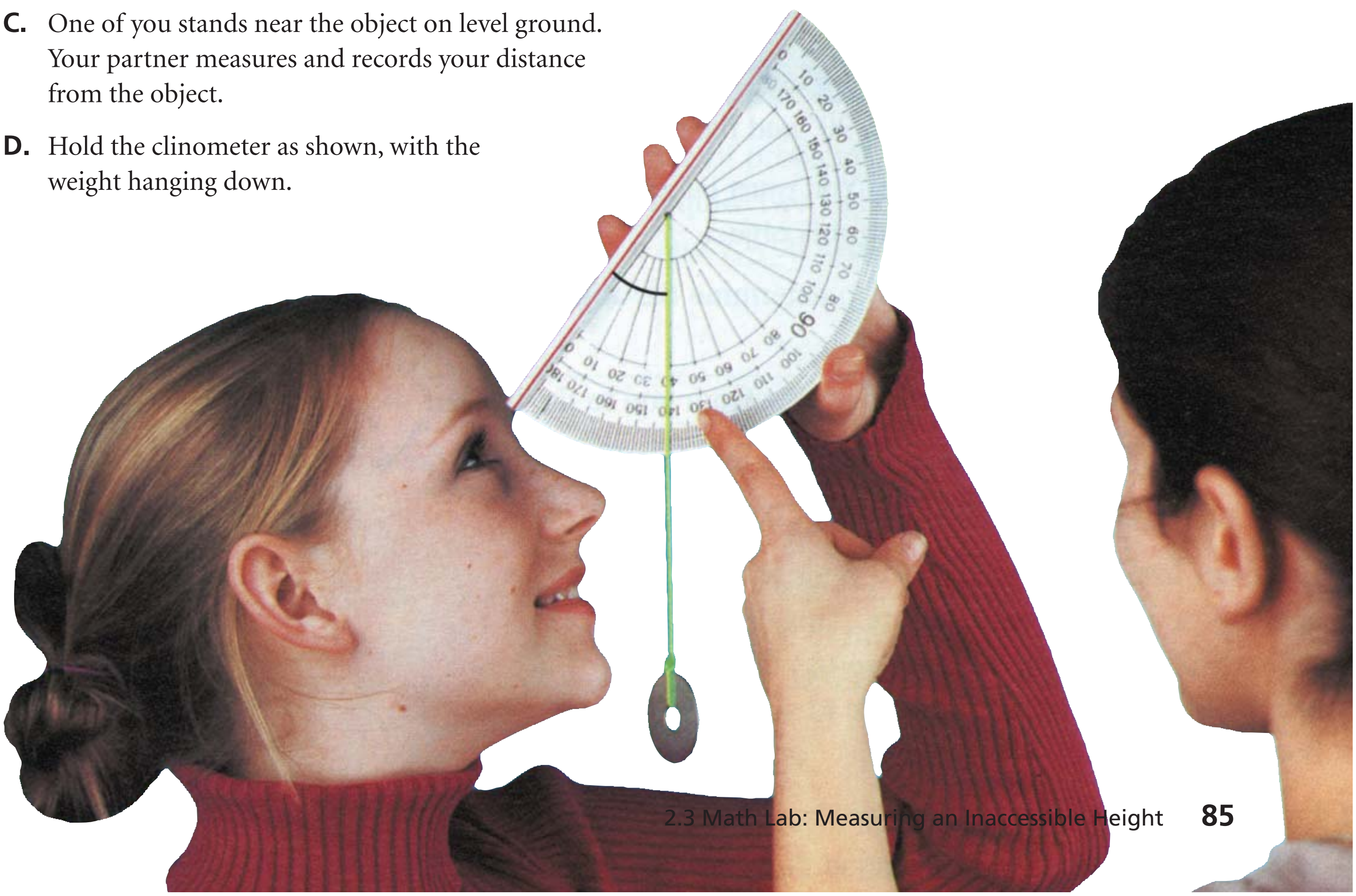
- Glue or tape the paper protractor to the cardboard. Carefully cut it out.
- Use the needle to pull the thread through the cardboard at the centre of baseline of the protractor. Secure the thread to the back of the cardboard with tape. Attach the weight to the other end of the thread.
- Tape the drinking straw along the baseline of the protractor for use as a sighting tube.



B. With your partner, choose a tall object whose height you cannot measure directly; for example, a flagpole, a totem pole, a tree, or a building.

C. One of you stands near the object on level ground. Your partner measures and records your distance from the object.

D. Hold the clinometer as shown, with the weight hanging down.



How does the acute angle between the thread and the straw relate to the angle of inclination of the straw?

E. Look at the top of the object through the straw. Your partner records the acute angle indicated by the thread on the protractor.

F. Your partner measures and records how far your eye is above the ground.

G. Sketch a diagram with a vertical line segment representing the object you want to measure. Label:

- your distance from the object
- the vertical distance from the ground to your eyes
- the angle of inclination of the straw

H. Change places with your partner. Repeat Steps B to G.

What other strategy could you use to determine the height of the object?

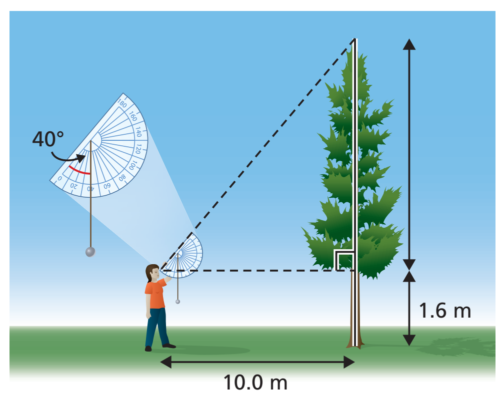
I. Use your measurements and the tangent ratio to calculate the height of the object.

J. Compare your results with those of your partner. Does the height of your eye affect the measurements? The final result? Explain.

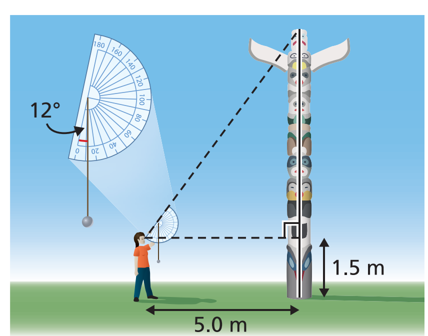
Assess Your Understanding

1. Explain how the angle shown on the protractor of your clinometer is related to the angle of inclination that the clinometer measures.

2. A tree farmer stood 10.0 m from the base of a tree. She used a clinometer to sight the top of the tree. The angle shown on the protractor scale was 40°. The tree farmer held the clinometer 1.6 m above the ground. Determine the height of the tree to the nearest tenth of a metre. The diagram is *not* drawn to scale.



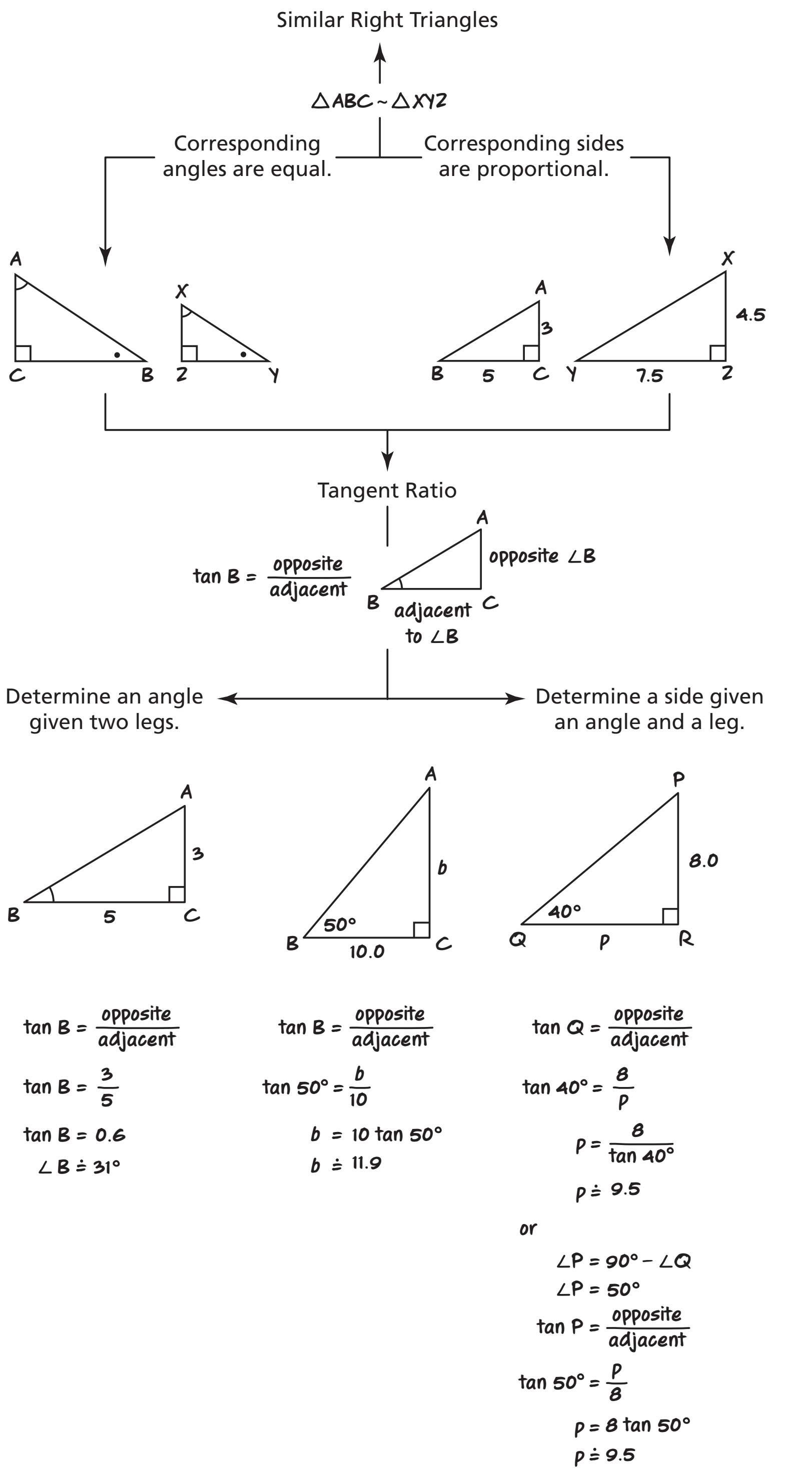
3. Use the information in the diagram to calculate the height of a totem pole observed with a drinking-straw clinometer. Give the answer to the nearest metre. The diagram is *not* drawn to scale.



Keep your clinometer for use in the Review.

CHECKPOINT 1

Connections



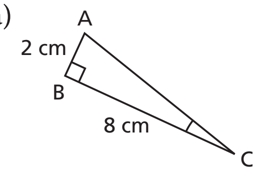
Concept Development

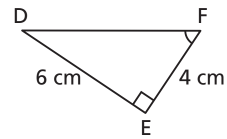
- **In Lesson 2.1**
  - You applied what you know about similar right triangles to develop the concept of the **tangent ratio**.
  - You used the tangent ratio **to determine an acute angle** in a right triangle when you know the lengths of the legs.
- **In Lesson 2.2**
  - You showed how to determine the **length of a leg** in a right triangle when you know the measures of an acute angle and the other leg.
- **In Lesson 2.3**
  - You applied the tangent ratio to a real-world measurement problem.

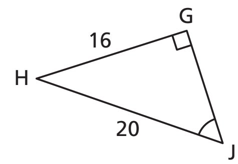
Assess Your Understanding

2.1

1. Determine the measure of each indicated angle to the nearest degree.

a)


b)


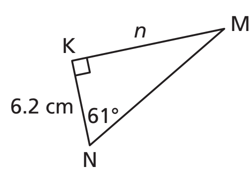
c)


2. Why does the tangent of an angle increase as the angle increases?

3. A small plane is flying at an altitude of 1000 m and is 5000 m from the beginning of the landing strip. What is the angle between the ground and the line of sight from an observer at the beginning of the landing strip? Give the measure to the nearest tenth of a degree.

2.2

4. Determine the length of each indicated side to the nearest tenth of a centimetre.

a)


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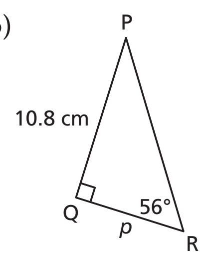
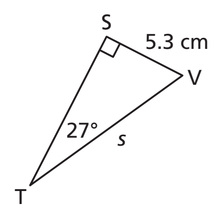


c)


5. A hiker saw a hoodoo on a cliff at Willow Creek in Alberta's badlands. The hiker was 9.1 m from the base of the cliff. From that point, the angle between the level ground and the line of sight to the top of the hoodoo was 69°. About how high was the top of the hoodoo above the level ground?

2.4 The Sine and Cosine Ratios

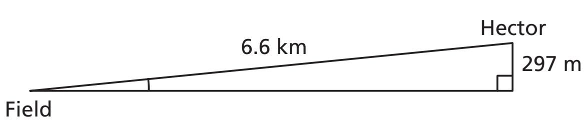
LESSON FOCUS

Develop and apply the sine and cosine ratios to determine angle measures.

Make Connections

The railroad track through the mountains between Field, B.C., and Hector, B.C., includes spiral tunnels. They were built in the early 1900s to reduce the angle of inclination of the track between the two towns. You can see a long train passing under itself after it comes out of a tunnel before it has finished going in.

Visualize the track straightened out to form the hypotenuse of a right triangle. Here is a diagram of the track before the tunnels were constructed. The diagram is *not* drawn to scale.



How could you determine the angle of inclination of the track?

Construct Understanding

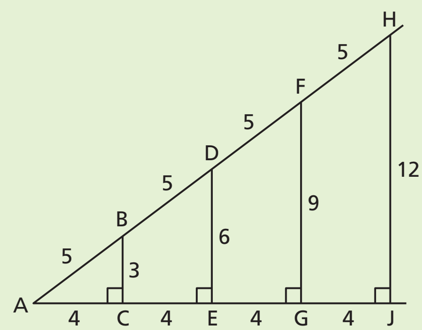
We defined the tangent ratio for an acute angle in a right triangle. There are two other ratios we can form to compare the sides of the triangle; each ratio involves the hypotenuse.

TRY THIS

Work with a partner.

You will need grid paper, a ruler, and a protractor.

A. Examine the nested right triangles below.



∠A is common to each triangle. How are the other acute angles in each triangle related? How do you know? How are the triangles related?

B. Copy and complete this table.

Triangle	Measures of Sides			Ratios	
	Hypotenuse	Side opposite ∠A	Side adjacent to ∠A	$\frac{\text{Side opposite } \angle A}{\text{Hypotenuse}}$	$\frac{\text{Side adjacent to } \angle A}{\text{Hypotenuse}}$
△ABC					
△ADE					
△AFG					
△AHJ					

C. Draw another set of nested right triangles that are not similar to those in Step A.

D. Measure the sides and angles of each triangle. Label your diagram with the measures, as in the diagram above.

E. Complete a table like the one in Step B for your triangles.

F. For each set of triangles, how do the ratios compare?

G. What do you think the value of each ratio depends on?

In a right triangle, the ratios that relate each leg to the hypotenuse depend only on the measure of the acute angle, and not on the size of the triangle. These ratios are called the **sine ratio** and the **cosine ratio**.

The sine ratio for ∠A is written as sin A and the cosine ratio for ∠A is written as cos A.

The Sine Ratio

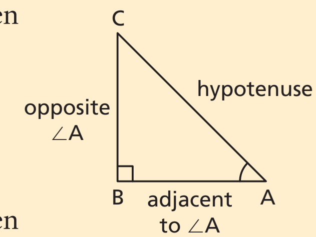
If ∠A is an acute angle in a right triangle, then

$$\sin A = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}}$$

The Cosine Ratio

If ∠A is an acute angle in a right triangle, then

$$\cos A = \frac{\text{length of side adjacent to } \angle A}{\text{length of hypotenuse}}$$



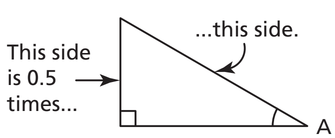
The tangent, sine, and cosine are called the **primary trigonometric ratios.** The word **trigonometry** comes from three Greek words "tri + gonia + metron" that together mean "three angle measure."

The branch of math that deals with the relations between the sides and angles of triangles is called **trigonometry**.

The values of the sine and cosine that compare the lengths of the sides are often expressed as decimals. For example, in right △ABC,

If sin A = 0.5, then in any similar right triangle, the length of the side opposite ∠A is 0.5 times the length of the hypotenuse.

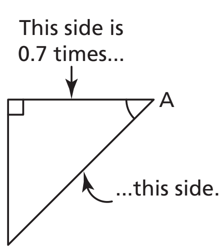
What happens to sin A as ∠A gets closer to 0°?



sin A = 0.5

If cos A = 0.7, then in any similar right triangle, the length of the side adjacent to ∠A is 0.7 times the length of the hypotenuse.

What happens to cos A as ∠A gets closer to 0°?

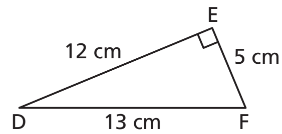


cos A = 0.7

Example 1 Determining the Sine and Cosine of an Angle

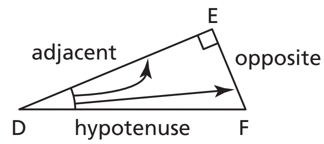
a) In △DEF, identify the side opposite ∠D and the side adjacent to ∠D.

b) Determine sin D and cos D to the nearest hundredth.



SOLUTION

a) In right △DEF,
DF is the hypotenuse.
EF is opposite ∠D and
DE is adjacent to ∠D.



b) $\sin D = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$\sin D = \dfrac{EF}{DF}$ — EF is opposite ∠D, DF is the hypotenuse.

$\sin D = \dfrac{5}{13}$

$\sin D = 0.3846\ldots$

$\sin D \doteq 0.38$

$\cos D = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

$\cos D = \dfrac{DE}{DF}$ — DE is adjacent to ∠D, DF is the hypotenuse.

$\cos D = \dfrac{12}{13}$

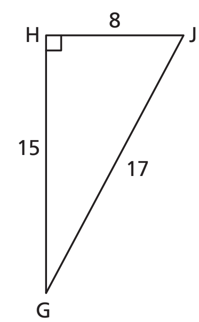
$\cos D = 0.9230\ldots$

$\cos D \doteq 0.92$

CHECK YOUR UNDERSTANDING

1. a) In △GHJ, identify the side opposite ∠G and the side adjacent to ∠G.

b) Determine sin G and cos G to the nearest hundredth.



[Answers: a) HJ, HG
b) sin G ≐ 0.47; cos G ≐ 0.88]

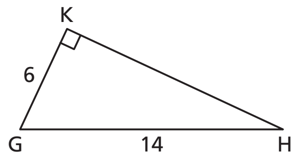
Determine sin F and cos F. How are these values related to sin D and cos D?

You can use a scientific calculator to determine the measure of an angle:

- When you know its sine, use $\sin^{-1}$ or InvSin
- When you know its cosine, use $\cos^{-1}$ or InvCos

Example 2 Using Sine or Cosine to Determine the Measure of an Angle

Determine the measures of ∠G and ∠H to the nearest tenth of a degree.



SOLUTIONS

Method 1

Determine the measure of ∠H first.

In right △GHK:

$\sin H = \frac{\text{opposite}}{\text{hypotenuse}}$ GK is opposite ∠H, GH is the hypotenuse.

$\sin H = \frac{GK}{GH}$

$\sin H = \frac{6}{14}$

$\angle H = 25.3769...°$

$\angle G + \angle H = 90°$

$\angle G = 90° - \angle H$

The angle sum of any triangle is 180°, so the two acute angles in a right triangle have a sum of 90°.

So, $\angle G = 90° - 25.3769...°$

$\angle G = 64.6230...°$

Method 2

Determine the measure of ∠G first.

In right △GHK:

$\cos G = \frac{\text{adjacent}}{\text{hypotenuse}}$ GK is adjacent to ∠G, GH is the hypotenuse.

$\cos G = \frac{GK}{GH}$

$\cos G = \frac{6}{14}$

$\angle G = 64.6230...°$

$\angle G + \angle H = 90°$ The two acute angles have a sum of 90°.

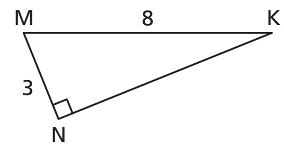
So, $\angle H = 90° - 64.6230...°$

$\angle H = 25.3769...°$

∠G is approximately 64.6° and ∠H is approximately 25.4°.

CHECK YOUR UNDERSTANDING

2. Determine the measures of ∠K and ∠M to the nearest tenth of a degree.



[Answer: ∠K ≐ 22.0°, ∠M ≐ 68.0°]

How are cos G and sin H related? Explain why this relationship occurs.

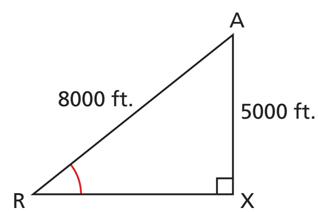
We can use the sine or cosine ratio to solve problems that can be modelled by a right triangle when we know the length of the hypotenuse, and the length of a leg or the measure of an acute angle.

Example 3 Using Sine or Cosine to Solve a Problem

A water bomber is flying at an altitude of 5000 ft. The plane's radar shows that it is 8000 ft. from the target site. What is the **angle of elevation** of the plane measured from the target site, to the nearest degree?

SOLUTION

Draw a diagram to represent the situation.
Altitude is measured vertically.
Assume the ground is horizontal.



∠R is the angle of elevation of the plane.

AX is the altitude of the plane.

RA is the distance from the target site to the plane.

In right △ARX:

$\sin R = \frac{AX}{RA}$

AX is opposite ∠R, RA is the hypotenuse.

$\sin R = \frac{5000}{8000}$

```
sin⁻¹(5000/8000)
        38.68218745
```

$\angle R \doteq 39°$

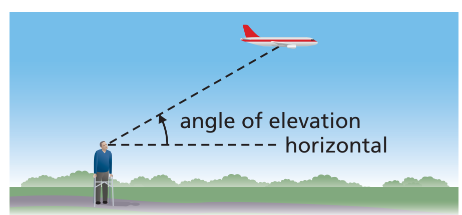
The angle of elevation of the plane is approximately 39°.

CHECK YOUR UNDERSTANDING

3. An observer is sitting on a dock watching a float plane in Vancouver harbour. At a certain time, the plane is 300 m above the water and 430 m from the observer. Determine the angle of elevation of the plane measured from the observer, to the nearest degree.

[Answer: approximately 44°]

The **angle of elevation** of an object above the horizontal is the angle between the horizontal and the line of sight from an observer.



Discuss the Ideas

1. When can you use the sine ratio to determine the measure of an acute angle in a right triangle? When can you use the cosine ratio?

2. Why is it important to draw a sketch before you start to solve a problem?

3. Why are the values of the sine of an acute angle and the cosine of an acute angle less than 1?

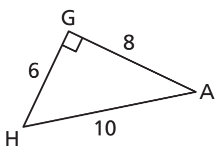
Exercises

A

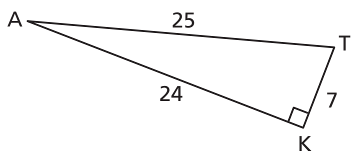
4. a) In each triangle below:

- Name the side opposite ∠A.
- Name the side adjacent to ∠A.
- Name the hypotenuse.

i)



ii)



b) For each triangle in part a, determine sin A and cos A to the nearest hundredth.

5. Determine the sine and cosine of each angle to the nearest hundredth.

a) 57° **b)** 5° **c)** 19° **d)** 81°

6. To the nearest degree, determine the measure of each ∠X.

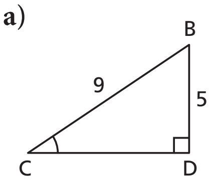
a) sin X = 0.25 **b)** cos X = 0.64

c) $\sin X = \frac{6}{11}$ **d)** $\cos X = \frac{7}{9}$

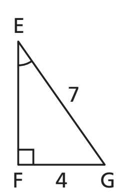
B

7. Determine the measure of each indicated angle to the nearest degree.

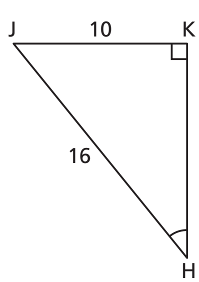
a)



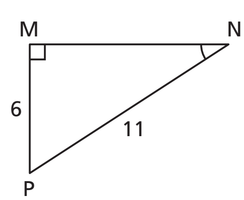
b)



c)

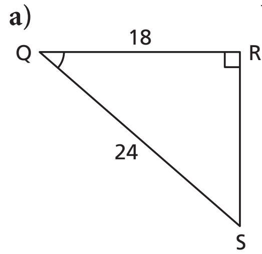


d)

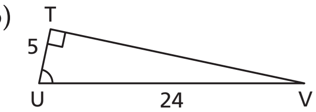


8. Determine the measure of each indicated angle to the nearest degree.

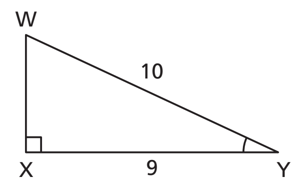
a)



b)



c)



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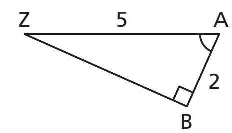


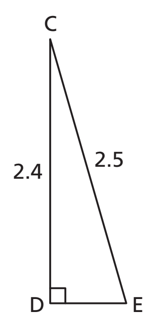
9. For each ratio below, sketch two different right triangles and label their sides.

a) $\sin B = \frac{3}{5}$ **b)** $\cos B = \frac{5}{8}$

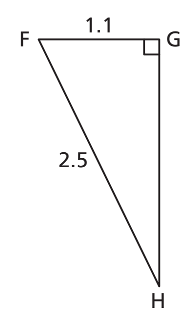
c) $\sin B = \frac{1}{4}$ **d)** $\cos B = \frac{4}{9}$

10. Use the sine or cosine ratio to determine the measure of each acute angle to the nearest tenth of a degree. Describe your strategy.

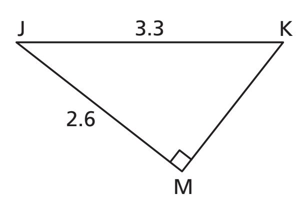
a)



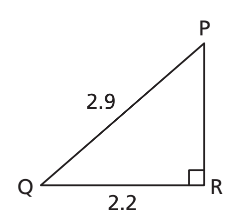
b)



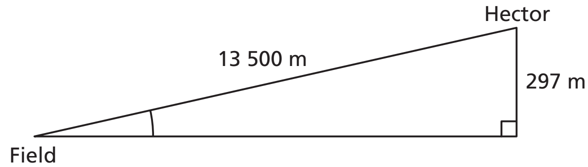
c)



d)



11. Suppose the railroad track through the spiral tunnels from Field to Hector were straightened out. It would look like the diagram below. The diagram is *not* drawn to scale. What is the angle of inclination of the track to the nearest tenth of a degree?



12. A ladder is 6.5 m long. It leans against a wall. The base of the ladder is 1.2 m from the wall. What is the angle of inclination of the ladder to the nearest tenth of a degree?

13. A rope that supports a tent is 2.4 m long. The rope is attached to the tent at a point that is 2.1 m above the ground. What is the angle of inclination of the rope to the nearest degree?

14. A rectangle is 4.8 cm long and each diagonal is 5.6 cm long. What is the measure of the angle between a diagonal and the longest side of the rectangle? Give the answer to the nearest degree.

15. **a)** Calculate:
i) sin 10° **ii)** sin 20° **iii)** sin 40°
iv) sin 50° **v)** sin 60° **vi)** sin 80°
b) Why does the sine of an angle increase as the angle increases?

16. Sketch a right isosceles triangle. Explain why the cosine of each acute angle is equal to the sine of the angle.

C

17. A cylindrical silo is 37 ft. high and has a diameter of 14 ft. The top of the silo can be reached by a spiral staircase that circles the silo once. What is the angle of inclination of the staircase to the nearest degree?

18. **a)** We have defined the sine and cosine ratios for acute angles. Use a calculator to determine:
i) sin 90° **ii)** sin 0° **iii)** cos 90° **iv)** cos 0°
b) Sketch a right triangle. Use the sketch to explain the results in part a.

Reflect

Explain when you might use the sine or cosine ratio instead of the tangent ratio to determine the measure of an acute angle in a right triangle. Include examples in your explanation.

THE WORLD OF MATH

Careers: Tool and Die Maker

A tool and die maker constructs tools and prepares dies for manufacturing common objects, such as bottle caps. A *die* is made up of two plates that stamp together. A tool and die maker uses trigonometry to construct a die. She works from blueprints that show the dimensions of the design. To cut the material for a die, a tool and die maker must set the milling machine at the precise angle.